

SEQUENCES OF SETS

A *sequence* is a function from the set $\mathbb{N} = \{1, 2, 3, \dots\}$ of positive integers into a set A . The notation a_n is used to denote the image of the integer n . Thus a sequence is usually denoted by

$$a_1, a_2, a_3, \dots$$

A finite sequence over a set A is a function from $\{1, 2, \dots, m\}$ into A . Such a finite sequence is called a *list*.

EXAMPLE

(a) The following are two familiar sequences:

$$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots \text{ which may be defined by } a_n = \frac{1}{n};$$

$$1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \dots \text{ which may be defined by } b_n = 2^{-n}$$

Note that the first sequence begins with $n = 1$ and the second sequence begins with $n = 0$.

(b) The sequence $1, -1, 1, -1, \dots$ may be defined by $a_n = (-1)^n$, where the sequence begins with $n = 0$.

Summation Symbol, Sums

Here we introduce the summation symbol $\sum$ (the Greek letter sigma). Consider a sequence $a_1, a_2, a_3, \dots$. Then we define the following:

$$\sum_{j=1}^n a_j = a_1 + a_2 + \dots + a_n$$

EXAMPLE :

$$\sum_{j=2}^5 j^2 = 2^2 + 3^2 + 4^2 + 5^2 = 4 + 9 + 16 + 25 = 54$$

$$\sum_{j=1}^n j = 1 + 2 + \dots + n = n(n+1)/2, \quad \text{for example, } 1 + 2 + \dots + 50 = (50 \times 51)/2 = 1275$$

RECURSIVELY DEFINED FUNCTIONS

A function is said to be *recursively defined* if the function definition refers to itself. In order for the definition not to be circular, the function definition must have the following two properties:

- (1) There must be certain arguments, called *base values*, for which the function does not refer to itself.
- (2) Each time the function does refer to itself, the argument of the function must be closer to a base value.

A recursive function with these two properties is said to be *well-defined*.

Factorial Function

The product of the positive integers from 1 to n , inclusive, is called " n factorial" and is usually denoted by $n!$. That is,

$$n! = n(n-1)(n-2) \cdot \cdot \cdot 3 \cdot 2 \cdot 1$$

where $0! = 1$, so that the function is defined for all nonnegative integers. Thus:

$$0! = 1,$$

$$1! = 1,$$

$$2! = 2 \cdot 1 = 2,$$

$$3! = 3 \cdot 2 \cdot 1 = 6,$$

$$4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$$

$$5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$$

$$6! = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 720$$

This is true for every positive integer n ; that is,

$$n! = n \cdot (n-1)!$$

Accordingly, the factorial function may also be defined as follows:

Definition of Factorial Function:

(a) If $n = 0$, then $n! = 1$.

(b) If $n > 0$, then $n! = n \cdot (n-1)!$

The definition of $n!$ is recursive, since it refers to itself when it uses $(n-1)!$. However:

(1) The value of $n!$ is explicitly given when $n = 0$ (thus 0 is a base value).

(2) The value of $n!$ for arbitrary n is defined in terms of a smaller value of n which is closer to the base value 0.

Accordingly, the definition is not circular, or, in other words, the function is well-defined.

EXAMPLE : the $4!$ Can be calculated in 9 steps using the recursive definition .

$$(1) \quad 4! = 4 \cdot 3!$$

$$(2) \quad 3! = 3 \cdot 2!$$

$$(3) \quad 2! = 2 \cdot 1!$$

$$(4) \quad 1! = 1 \cdot 0!$$

$$(5) \quad 0! = 1$$

$$(6) \quad 1! = 1 \cdot 1 = 1$$

$$(7) \quad 2! = 2 \cdot 1 = 2$$

$$(8) \quad 3! = 3 \cdot 2 = 6$$

$$(9) \quad 4! = 4 \cdot 6 = 24$$

Fibonacci Sequence

The Fibonacci sequence (usually denoted by $F_0, F_1, F_2, \dots$) is as follows:

$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \dots$$

That is, $F_0 = 0$ and $F_1 = 1$ and each succeeding term is the sum of the two preceding terms. For example, the next two terms of the sequence are

$$34 + 55 = 89 \text{ and } 55 + 89 = 144$$

Fibonacci Sequence can be defined

- (a) If $n = 0$, or $n = 1$, then $F_n = n$.
- (b) If $n > 1$, then $F_n = F_{n-2} + F_{n-1}$.

Where : The base values are 0 and 1, and the value of F_n is defined in terms of smaller values of n which are closer to the base values.

Accordingly, this function is well-defined.

Logic and propositional calculus

A *proposition* (or *statement*) is a declarative statement which is true or false, but not both.

Example: the following six sentences:

- (1) Ice floats in water.
- (2) China is in Europe.
- (3) $2 + 2 = 4$
- (4) $2 + 2 = 5$
- (5) Where are you going?
- (6) Do your homework.

The first four are propositions, the last two are not. Also, (1) and (3) are true, but (2) and (4) are false.

Compound Propositions

It is the proposition that composed of subpropositions and various connectives.

Primitive proposition is the proposition that cannot be broken down into simpler propositions.

For example, the above propositions are primitive propositions, while:

“Roses are red and violets are blue.” and

“John is smart or he studies every night.”, Are compound.

BASIC LOGICAL OPERATIONS

1- Conjunction, $p \wedge q$

2- Disjunction, $p \vee q$

3- Negation, $\neg p$

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

(a) “ p and q ”

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

(b) “ p or q ”

p	$\neg p$
T	F
F	T

(c) “not p ”

EXAMPLE : Consider the following four statements:

- (i) Ice floats in water **and** $2 + 2 = 4$.
- (ii) Ice floats in water **and** $2 + 2 = 5$.
- (iii) China is in Europe **and** $2 + 2 = 4$.
- (iv) China is in Europe **and** $2 + 2 = 5$.

Only the first statement is true. Each of the others is false since at least one of its substatements is false.

EXAMPLE: Consider the following four statements:

- (i) Ice floats in water **or** $2 + 2 = 4$.
- (ii) Ice floats in water **or** $2 + 2 = 5$.
- (iii) China is in Europe **or** $2 + 2 = 4$.
- (iv) China is in Europe **or** $2 + 2 = 5$.

Only the last statement (iv) is false. Each of the others is true since at least one of its sub-statements is true.

EXAMPLE : Consider the following six statements:

(a1) Ice floats in water. (a2) It is false that ice floats in water. (a3) Ice does not float in water.
(b1) $2 + 2 = 5$ (b2) It is false that $2 + 2 = 5$. (b3) $2 + 2 \neq 5$

Then (a2) and (a3) are each the negation of (a1); and (b2) and (b3) are each the negation of (b1). Since (a1) is true, (a2) and (a3) are false; and since (b1) is false, (b2) and (b3) are true.

PROPOSITIONS AND TRUTH TABLES

The truth table for the compound proposition $\neg(p \wedge \neg q)$ is:

p	q	$\neg q$	$p \wedge \neg q$	$\neg(p \wedge \neg q)$
T	T	F	F	T
T	F	T	T	F
F	T	F	F	T
F	F	T	F	T

TAUTOLOGIES AND CONTRADICTIONS

Some propositions $P(p, q, \dots)$ contain only T in the last column of their truth tables or, in other words, they are true for any truth values of their variables. Such propositions are called tautologies. Analogously, a proposition $P(p, q, \dots)$ is called a contradiction if it contains only F in the last column of its truth table or, in other words, if it is false for any truth values of its variables. For example, the proposition “p or not p,” that is, $p \vee \neg p$, is a tautology, and the proposition “p and not p,” that is, $p \wedge \neg p$, is a contradiction.

p	$\neg p$	$p \vee \neg p$
T	F	T
F	T	T

(a) $p \vee \neg p$

p	$\neg p$	$p \wedge \neg p$
T	F	F
F	T	F

(b) $p \wedge \neg p$

Note that the negation of a tautology is a contradiction since it is always false, and the negation of a contradiction is a tautology since it is always true.

LOGICAL EQUIVALENCE

Two propositions $P(p, q, \dots)$ and $Q(p, q, \dots)$ are said to be logically equivalent, or equal, denoted by $P(p, q, \dots) \equiv Q(p, q, \dots)$ if they have identical truth tables.

p	q	$p \wedge q$	$\neg(p \wedge q)$
T	T	T	F
T	F	F	T
F	T	F	T
F	F	F	T

(a) $\neg(p \wedge q)$

p	q	$\neg p$	$\neg q$	$\neg p \vee \neg q$
T	T	F	F	F
T	F	F	T	T
F	T	T	F	T
F	F	T	T	T

(b) $\neg p \vee \neg q$

for example, the truth tables of $\neg(p \wedge q)$ and $\neg p \vee \neg q$, both truth tables are the same, that is, both propositions are false in the first case and true in the other three cases. Accordingly, we can write:

$$\neg(p \wedge q) \equiv \neg p \vee \neg q$$

ALGEBRA OF PROPOSITIONS

Idempotent laws:	(1a) $p \vee p \equiv p$	(1b) $p \wedge p \equiv p$
Associative laws:	(2a) $(p \vee q) \vee r \equiv p \vee (q \vee r)$	(2b) $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$
Commutative laws:	(3a) $p \vee q \equiv q \vee p$	(3b) $p \wedge q \equiv q \wedge p$
Distributive laws:	(4a) $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$	(4b) $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$
Identity laws:	(5a) $p \vee F \equiv p$ (6a) $p \vee T \equiv T$	(5b) $p \wedge T \equiv p$ (6b) $p \wedge F \equiv F$
Involution law:	(7) $\neg\neg p \equiv p$	
Complement laws:	(8a) $p \vee \neg p \equiv T$ (9a) $\neg T \equiv F$	(8b) $p \wedge \neg p \equiv F$ (9b) $\neg F \equiv T$
DeMorgan's laws:	(10a) $\neg(p \vee q) \equiv \neg p \wedge \neg q$	(10b) $\neg(p \wedge q) \equiv \neg p \vee \neg q$

CONDITIONAL AND BICONDITIONAL STATEMENTS

Many statements, in mathematics, are of the form "If p then q." Such statements are called conditional statements and are denoted by: $p \rightarrow q$

The conditional $p \rightarrow q$ is frequently read "p implies q" or "p only if q."

Another statement is of the form "p if and only if q." Such statements are called biconditional statements and are denoted by $p \leftrightarrow q$

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

(a) $p \rightarrow q$

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

(b) $p \leftrightarrow q$

(a) The conditional $p \rightarrow q$ is false only when the first part p is true and the second part q is false. Accordingly, when p is false, the conditional $p \rightarrow q$ is true regardless of the truth value of q .

(b) The biconditional $p \leftrightarrow q$ is true whenever p and q have the same truth values and false otherwise.

Note that the truth table of $p \rightarrow q$ and $\neg p \vee q$ are identical, that is, they are both false only in the second case. Accordingly, $p \rightarrow q$ is logically equivalent to $\neg p \vee q$; that is,

$$p \rightarrow q \equiv \neg p \vee q$$

p	q	$\neg p$	$\neg p \vee q$	p	q	$p \rightarrow q$
T	T	F	T	T	T	T
T	F	F	F	T	F	F
F	T	T	T	F	T	T
F	F	T	T	F	F	T

(c) $\neg p \vee q$

(a) $p \rightarrow q$

Arguments

A **valid argument** is a finite set of propositions $P_1, \dots, P_r$ called premises, together with a proposition C , the conclusion, such that the propositional form $(P_1 \wedge P_2 \wedge \dots \wedge P_r) \rightarrow C$ is a tautology.

We say C is a logical consequence of the premises.

We write $P_1, \dots, P_r \vdash C$. The symbol $\vdash$ is called the turnstile.

If an argument is not valid we say that it is invalid.

Method to prove validity

Example :

Let Q_1 = "John graduates"

Q_2 = "Mary graduates"

Q_3 = "John gets a job"

Q_4 = "Mary gets a job"

Q_5 = "Mary earns money"

(i) Consider the following argument:

"If John graduates then he gets a job".

"John graduates".

"Therefore John gets a job".

To see the "form" of this argument we symbolize it as $Q_1 \rightarrow Q_3, Q_1 \vdash Q_3$.

Now we check that $((Q_1 \rightarrow Q_3) \wedge Q_1) \rightarrow Q_3$ is a tautology:

Q_1	Q_3	$Q_1 \rightarrow Q_3$	$(Q_1 \rightarrow Q_3) \wedge Q_1$	$((Q_1 \rightarrow Q_3) \wedge Q_1) \rightarrow Q_3$
T	T	T	T	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	T

This tautology shows that the argument is valid.

Example (ii) Consider the following argument:

“If Mary graduates then she gets a job”.

“Mary does not get a job”.

“Therefore Mary does not graduate”.

Symbolized, this becomes: $Q_2 \rightarrow Q_4, (\neg Q_4) \vdash (\neg Q_2)$.

Now check that $((Q_2 \rightarrow Q_4) \wedge (\neg Q_4)) \vdash (\neg Q_2)$ is a tautology.

Q_2	Q_4	$Q_2 \rightarrow Q_4$	$\neg Q_4$	$(Q_2 \rightarrow Q_4) \wedge (\neg Q_4) (\equiv A)$	$\neg Q_2$	$A \rightarrow (\neg Q_2)$
T	T	T	F	F	F	T
T	F	F	T	F	F	T
F	T	T	F	F	T	T
F	F	T	T	T	T	T

Example (iii) Consider the following argument:

“Either Mary or John graduate”.

“John does not graduate”.

“Therefore Mary graduates”.

Symbolized, this becomes $Q_2 \vee Q_1, (\neg Q_1) \vdash Q_2$.

*Now check that $((Q_2 \vee Q_1) \wedge (\neg Q_1)) \rightarrow Q_2$ is a tautology.

Q_1	Q_2	$Q_2 \vee Q_1$	$\neg Q_1$	$(Q_2 \vee Q_1) \wedge (\neg Q_1) (\equiv B)$	$B \rightarrow Q_2$
T	T	T	F	F	T
T	F	T	F	F	T
F	T	T	T	T	T
F	F	F	T	F	T

Example (iv) Consider the following argument:

“If Mary graduates then she gets a job”.

“If Mary gets a job then she earns money”.

“Therefore if Mary graduates then she earns money”.

Symbolized, this becomes $Q_2 \rightarrow Q_4, Q_4 \rightarrow Q_5 \vdash (Q_2 \rightarrow Q_5)$.

*Now check that $((Q_2 \rightarrow Q_4) \wedge (Q_4 \rightarrow Q_5)) \rightarrow (Q_2 \rightarrow Q_5)$ is a tautology.

Q_2	Q_4	Q_5	$Q_2 \rightarrow Q_4$ A	$Q_4 \rightarrow Q_5$ B	$A \wedge B$	$Q_2 \rightarrow Q_5$ C	$(A \wedge B) \rightarrow C$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	T
T	F	T	F	T	F	T	T
T	F	F	F	T	F	F	T
F	T	T	T	T	T	T	T
F	T	F	T	F	F	T	T
F	F	T	T	T	T	T	T
F	F	F	T	T	T	T	T

We can sum up the above by saying the following are all valid:

- (i) $p \rightarrow q, p \vdash q$,
- (ii) $p \rightarrow q, \neg q \vdash \neg p$,
- (iii) $p \vee q, \neg q \vdash p$,
- (iv) $p \rightarrow q, q \rightarrow r \vdash p \rightarrow r$.

Example Show that $p \rightarrow q, q \vee p \vdash (\neg q) \vee (\neg p)$ is valid.

p	q	$p \rightarrow q$	$q \vee p$	$((p \rightarrow q) \wedge (q \vee p))$ P	$((\neg q) \vee (\neg p))$ C	$P \rightarrow C$	
T	T	T	T	T	F	F	←
T	F	F	T	F	T	T	
F	T	T	T	T	T	T	
F	F	T	F	F	T	T	

We do not have a tautology in the last column so the argument is invalid.

Homework:

Is $p \rightarrow q, q \vee p \vdash (\neg q) \vee (\neg p)$ valid?

Is $p \rightarrow (s \rightarrow (\neg r)), p \rightarrow r, p \vdash \neg s$ valid?

Is $(p \vee q) \rightarrow s, q \rightarrow s \vdash s$ valid?